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INTERPOLATION SETS FOR SUBALGEBRAS OF $l^\infty(\mathbf{Z})$

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ABSTRACT

Let $\mathfrak{A}$ be the subalgebra of $l^\infty(\mathbf{Z})$ generated by the minimal functions. The collection of $\mathfrak{A}$-interpolation sets is identified as the ideal of small subsets of $\mathbf{Z}$. General theorems about the relation between invariant ideals and collections of $\mathcal{A}$-interpolation sets, for subalgebras $\mathcal{A}$ of l^∞, are proven.

§1. Introduction

An ideal of subsets of the integers $\mathbf{Z}$ is a collection of subsets $\mathcal{I}$ such that (i) A, $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$, and (iii) $\mathbf{Z} \notin \mathcal{I}$. The ideal is said to be invariant if $A \in \mathcal{I}$ iff $A + 1 = \{a + 1 : a \in A\} \in \mathcal{I}$.

For example, the collections of finite sets, Sidon sets, and sets of density zero all form invariant ideals. The collection of sets A such that $A \supset B - B$ implies that B is either finite or empty, is an ideal (use Ramsey's theorem) which is not invariant.

Let $\mathcal{F}$ be the subalgebra of $l^\infty(\mathbf{Z})$ which consists of all Fourier transforms of measures on the circle. A subset $A \subset \mathbf{Z}$ is Sidon, if every bounded function on A can be extended to a function in $\mathcal{F}$; i.e. if A is an interpolation set for $\mathcal{F}$. The question, whether the collection $\mathcal{I}_{\mathcal{F}}$ of $\mathcal{F}$-interpolation sets (Sidon sets) is an ideal, was answered affirmatively by Drury (1970). It is also known that one can replace $\mathcal{F}$ by its uniform closure in $l^\infty(\mathbf{Z})$, when defining Sidon sets, i.e. $\mathcal{I}_{\mathcal{F}} = \mathcal{I}_{\bar{\mathcal{F}}}$.

If one considers, on the other hand, the algebra $\mathcal{E} \subset l^\infty(\mathbf{Z})$ of almost periodic functions and the corresponding collection $\mathcal{I}_{\mathcal{E}}$ of $\mathcal{E}$-interpolation sets, it is then an easy result that this collection is not an ideal. In the positive direction again, it is shown in [6] that for the algebra $\tilde{\mathcal{E}}$ of almost automorphic functions, $\mathcal{I}_{\tilde{\mathcal{E}}}$ is an ideal.

Given a norm closed translation invariant subalgebra $\mathcal{A}$ of $l^\infty(\mathbf{Z})$ containing the constant functions (in brief, an algebra), we say that a subset $A \subset \mathbf{Z}$ is an

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$\mathscr{A}$*-interpolation set* if every bounded real valued function on A can be extended to a function in $\mathscr{A}$. We write $\mathscr{I}_{\mathscr{A}}$ for the collection of all $\mathscr{A}$-interpolation sets.

The types of questions in which we are interested here are:

(1) Given $\mathscr{A}$, is $\mathscr{I}_{\mathscr{A}}$ an ideal?

(2) Given $\mathscr{A}$, characterize $\mathscr{I}_{\mathscr{A}}$.

(3) Given an invariant ideal $\mathscr{I}$, is there an algebra $\mathscr{A}$ for which $\mathscr{I} = \mathscr{I}_{\mathscr{A}}$?

For previous work in this direction, we refer the reader to [6]. We now proceed to describe our main results.

A function $f \in l^\infty(\mathbf{Z}) = l^\infty$ is called *minimal* if $\forall \varepsilon > 0$, $\forall F$ a finite subset of $\mathbf{Z}$ the set

$$\{n \in \mathbf{Z} : |f(n+i) - f(i)| < \varepsilon, \forall i \in F\}$$

is syndetic (i.e. has bounded gaps). (An equivalent condition is that the closure of the set of translates of f in the topology of pointwise convergence forms a minimal set.)

The algebras $\mathscr{E}$ and $\tilde{\mathscr{E}}$ of almost periodic and almost automorphic functions consist of minimal functions. Unlike those algebras the set $\mathscr{M} \subset l^\infty(\mathbf{Z})$ of all minimal functions in l^∞ does not form an algebra (neither sums nor products of minimal functions need be minimal). However, the family of maximal algebras of minimal functions can be parametrized by $v \in J$, where J is the set of idempotents in a fixed minimal left ideal of $\beta\mathbf{Z}$, the Stone–Čech compactification of $\mathbf{Z}$. We denote this family by $\{\mathfrak{A}(v)\}_{v \in J}$ and note that all algebras in this family are isometrically isomorphic. The smallest algebra containing $\mathscr{M}$, or equivalently the algebra generated by all the $\mathfrak{A}(v)$, will be denoted by $\mathfrak{A}$. We write 1_A for the characteristic function of a set A. In particular $1_\varnothing$ is the function 0.

A subset $A \subset Z$ is *small* if $\{1_\varnothing\}$ is the unique minimal set in $\bar{\mathscr{O}}(1_A)$ in the flow (Ω_2, σ) where $\Omega_2 = \{0,1\}^Z$, σ is the shift on Ω_2 and $\bar{\mathscr{O}}$ stands for orbit closure. Equivalently A is small if for every $k > 0$ there exists an $N_k > 0$ such that in every interval of length N_k in $\mathbf{Z}$ there are k consecutive members of A^c, the complement of A.

THEOREM 1. (1) *Every set in* $\mathscr{I}_{\mathfrak{A}}$ *is small.* (2) *Given a small set* $A \subset Z$, *there exists an algebra* $\mathscr{A}$ *of minimal functions such that* $A \in \mathscr{I}_{\mathscr{A}}$; *thus,* (3) $\mathscr{I}_{\mathfrak{A}}$ *coincides with the ideal of small sets. In particular* $\mathfrak{A} \neq l^\infty$.

The last assertion answers an old problem of Furstenberg. In [2] he was dealing with a restricted version of this problem to symbolic flows and conjectured that a similar situation presents itself in the general case. We will use his results in our proof of Theorem 1 (1). We do not know whether for a given

$v \in J$ the collection $\mathscr{I}_{\mathfrak{A}(v)}$ is an ideal. The statement in [6; th. 5.1. (7)] "that $\mathscr{I}_{\mathfrak{A}(v)}$ is not an ideal follows from Veech's work [8]" is erroneous.

If $\mathscr{I}$ is an ideal of subsets of $\mathbf{Z}$ then $\mathscr{F} = \{A^c : A \in \mathscr{I}\}$ is a filter. Thus the set $K = \bigcap \{\overline{A^c} : A \in \mathscr{I}\}$, where $\bar{B}$, for $B \subset \mathbf{Z}$, denotes the closure of B in $\beta\mathbf{Z}$, is a non-empty closed subset of $\beta\mathbf{Z}$. $\mathscr{I}$ is *free* if K does not contain points of $\mathbf{Z}$ ($\Leftrightarrow \cap \mathscr{F} = \emptyset$). An invariant ideal is clearly free. (In [6] the collection $\mathscr{P} = \{A \subset \mathbf{Z} : A \notin \mathscr{I}\}$ was called a divisible property and the set K, the kernel of $\mathscr{P}$.)

Given an invariant ideal $\mathscr{I}$ we let $\mathfrak{B}(K)$ be the algebra of functions $f \in l^\infty$ whose extension to $\beta\mathbf{Z}$ does not distinguish between points of K.

THEOREM 2. (1) *Let $\mathscr{I}$ be an invariant ideal, then the following are equivalent*:

(a) $\mathscr{I} = \mathscr{I}_{\mathfrak{B}(K)}$.

(b) *K does not contain isolated points.*

(2) *In any case $\mathscr{I} \subset \mathscr{I}_{\mathfrak{B}(K)}$ and if they are not equal then for no algebra $\mathscr{A}$ is it true that $\mathscr{I} = \mathscr{I}_{\mathscr{A}}$.*

(3) *There exists an invariant ideal for which K has an isolated point.*

In several concrete examples the condition (b) can be directly verified and thus we obtain

COROLLARY. *Each of the following invariant ideals is the ideal of interpolation sets of some algebra*:

(1) *The ideal of small sets.*

(2) *The ideal of sets $A \subset \mathbf{Z}$ with $\bar{A} \cap M \neq \emptyset$, where M is a fixed minimal left ideal in $\beta\mathbf{Z}$.*

(3) *The ideal of sets $A \subset \mathbf{Z}$ with uniform zero density, i.e.*

$$\lim_{k \to \infty} \left\| \frac{1}{k} \sum_{1}^{k} 1_{A+j} \right\|_\infty = 0.$$

(In the terminology of [3] this property is called "upper Banach density zero".)

We remark that an ideal $\mathscr{I}$ can equal $\mathscr{I}_{\mathscr{A}}$ for many different algebras $\mathscr{A}$; e.g., for every separable algebra $\mathscr{A}$, $\mathscr{I}_{\mathscr{A}}$ is the collection of finite sets. Another example is the equality {small sets} $= \mathscr{I}_{\mathfrak{A}} = \mathscr{I}_{\mathfrak{B}(K)}$ where $K = \{\overline{\bigcup M} : M$ minimal ideal in $\beta\mathbf{Z}\}$.

Given a pointed flow (X, x_0, T), with $\bar{\mathcal{O}}(x_0) = X$, the mapping $F \to f$ of $C(X)$ into $l^\infty(\mathbf{Z})$ given by $f(n) = F(T^n x_0)$ is an isometric isomorphism. Conversely, starting with an algebra $\mathscr{A} \subset l^\infty$, there is a pointed flow $(|\mathscr{A}|, x_0, T)$ with $C(|\mathscr{A}|) \cong \mathscr{A}$ under the above correspondence ($|\mathscr{A}|$ is the space of multiplicative continuous functionals on $\mathscr{A}$; $x_0 : \mathscr{A} \to R$ is defined by $x_0(f) = f(0)$). Let $\mathscr{A}$ be an algebra and put

$$\mathscr{F}(\mathscr{A}) = \{N(x_0, V) : V \text{ is a neighborhood of } x_0 \text{ in } |\mathscr{A}|\}$$

where $N(x_0, V) = \{n : T^n x_0 \in V\}$. Let

$$K_0 = \bigcap \{\bar{F} : F \in \mathscr{F}(\mathscr{A})\} \setminus \{0\}.$$

Notice that when $|\mathscr{A}|$ is infinite

$$K_0 = \{p \in \beta \mathbf{Z} \setminus \mathbf{Z} : px_0 = x_0\}$$

and $K_0 = \emptyset$ iff $\mathscr{A} = l^\infty$.

Put $\mathscr{S}(\mathscr{A}) = \{A \subset \mathbf{Z} : \forall n \in \mathbf{Z},\ \overline{A + n} \cap K_0 = \emptyset\}$. We recall, [6], that $J_{\mathscr{A}}$ is the topology on $\mathbf{Z}$ induced by the embedding $n \to T^n x_0$ of $\mathbf{Z}$ into $|\mathscr{A}|$ and that $\tilde{\mathscr{A}}$ is the algebra of all $J_{\mathscr{A}}$ continuous functions in l^∞. We have $\tilde{\mathscr{A}} \supset \mathscr{A}$ and $\tilde{\tilde{\mathscr{A}}} = \tilde{\mathscr{A}}$ thus $J_{\mathscr{A}} = J_{\tilde{\mathscr{A}}}$ and $\mathscr{S}(\mathscr{A}) = \mathscr{S}(\tilde{\mathscr{A}})$.

THEOREM 3. (1) *$A \in \mathscr{S}(\mathscr{A})$ iff A is $J_{\mathscr{A}}$ closed and discrete.*

(2) *$\mathscr{S}(\mathscr{A}) \subset \mathscr{I}_{\tilde{\mathscr{A}}}$.*

(3) *$\mathscr{S}(\mathscr{A}) \supset \mathscr{I}_{\mathscr{A}}$ in each of the following cases:*

(i) *$\mathscr{A}$ as a subset of the Polonais space $\mathbf{R}^{\mathbf{Z}}$ is Souslin.*

(ii) *The collection $\mathscr{I}_{\mathscr{A}}$ is closed under union with finite sets.*

(iii) *K_0 has no isolated points.*

(4) *Under any of the conditions in* (3) *(with $\tilde{\mathscr{A}}$ replacing $\mathscr{A}$) $\mathscr{I}_{\tilde{\mathscr{A}}} = \mathscr{S}(\tilde{\mathscr{A}})$.*

Notice that by [8], $\mathscr{A}$ is Souslin iff $\tilde{\mathscr{A}}$ is Souslin. In checking case (3)(i) it is sometimes useful to consider the nature of $J_{\mathscr{A}}$ as a subset of Ω_2. Our last result is connected with this question.

A theorem of Sierpinski [7] implies that the collection of subsets of $\mathbf{Z}$ corresponding to neighborhoods of a point in $\beta\mathbf{Z} \setminus \mathbf{Z}$, i.e. a free ultrafilter on $\mathbf{Z}$, is not a Souslin subset of $2^{\mathbf{Z}} = \Omega_2$. In [8; 3.2.] Veech asks what can be said in this respect about the collection of subsets of $\mathbf{Z}$ which are neighborhoods of some minimal idempotent (or just idempotent) in $\beta\mathbf{Z} \setminus \mathbf{Z}$.

The notion of an IP-set was introduced in [4] (see also [3]), and it was shown in [6] that a subset A of $\mathbf{Z}$ contains an IP-set iff $\bar{A}$ is a neighborhood of an idempotent in $\beta\mathbf{Z} \setminus \mathbf{Z}$. We say that A is an MIP-set if $\bar{A}$ contains a minimal idempotent. It follows from [6] that A is MIP iff it is central in the terminology of [3] (see definitions below). We use these characterizations to prove Theorem 4.

THEOREM 4. (1) *The collection of* IP-*sets considered as a subset of* Ω_2 *is Souslin.*

(2) *The collection of* MIP-*sets is Souslin.*

In section 2, we prove Theorem 1. Theorems 2, 3 and 4 are proven in section 3. We also give there an example of an algebra $\mathcal{A} \subset l^\infty$ for which $\mathcal{I}_{\mathcal{A}}$ has the property that $\mathbf{Z}$ is the union of a finite number of sets in $\mathcal{I}_{\mathcal{A}}$. The question whether such an algebra exists was posed in [6].

§2. Small sets

Let $\Omega_2 = \{0,1\}^Z$, we consider Ω_2 both as a flow under the shift and as a compact topological ring under coordinate-wise multiplication and addition modulo 2. The following definitions and results are from [2].

A closed shift invariant subset X of Ω_2 is *restricted* if $X + Y = \Omega_2$ for closed invariant Y implies $Y = \Omega_2$. Minimal subsets of (Ω_2, σ) are restricted; moreover if M is minimal and X is restricted, then MX is restricted. Clearly, every finite sum of restricted sets is restricted, and we conclude that $Z = \sum_{j=1}^{m} M_{j1}M_{j2}\cdots M_{jk_j}$ is restricted whenever the M_{jl}'s are minimal sets. Let R be the union of all restricted subsets of Ω_2, then R contains all restricted sets and, in fact, any closed invariant subset of R is restricted.

Let R_0 be the subring of Ω_2 generated by the minimal functions in Ω_2 (i.e. $\mathcal{M} \cap \Omega_2$, see Lemma 2.4). Then $R_0 \subset R$ and for every finite number of symbolic minimal flows X_i $(i = 1, \cdots, n)$, $x_0 = (x_1, \cdots, x_n) \in \prod_{i=1}^{n} X_i$ and a $\{0,1\}$-valued, continuous function F on X the function $f(n) = F(T^n x_0)$ is in R_0. (The latter statement follows from the simple observation that a $\{0,1\}$-valued continuous function f on a symbolic flow depends only on a finite number of coordinates, say ξ_i, $|i| \leqq N$ and has the form

$$f(\xi_{-N}, \cdots, \xi_N) = \sum_J a_J \prod_{j \in J} \xi_j$$

where the a_J's are 0 or 1, the summation is modulo 2 and J ranges over the subsets of $\{-N, \cdots, N\}$.)

We shall use these results in the proof of Theorem 1(1); however, we first need some lemmas about $\mathfrak{A}$.

LEMMA 2.1. *If* $f \in \mathfrak{A}$ *then any pointwise limit of the form* $\lim T^{n_i} f = g$ *is also in* $\mathfrak{A}$ $(Tf(k) = f(k+1))$.

PROOF. Since $f \in \mathfrak{A}$ it can be approximated uniformly by linear combinations of the form $\Sigma_{j=1}^{m} f_{j1}, \cdots, f_{jk_j}$ where the f_{jl} are minimal functions. Each pointwise limit of the form $\lim T^{n_i} f_{jl}$ is minimal and our lemma follows. □

We recall that the pointed product of two pointed flows $(X, x_0) \vee (Y, y_0)$ is the subflow $\bar{\mathcal{O}}(x_0, y_0) \subset X \times Y$. Likewise if (X_i, x_i) is any family of pointed flows then $\vee (X_i, x_i) = (X, x)$ is the orbit closure of $x \in \Pi X_i$ where x is the point whose i-th coordinate is x_i. If $\mathcal{A}_i$ is the algebra corresponding to (X_i, x_i), then the algebra $\mathcal{A}$ which corresponds to (X, x) is the smallest algebra containing all the $\mathcal{A}_i$, i.e. $\mathcal{A} = \vee \mathcal{A}_i$ and $|\mathcal{A}| = X$.

LEMMA 2.2. *Let (X, x_0) be a pointed metric minimal flow, then there exist pointed metric minimal flows $(\tilde{X}, \tilde{x}_0)$ and (Z_n, z_n) where each Z_n is a subflow of Ω_2, such that $(\tilde{X}, \tilde{x}_0) = \vee_{n \in \mathbf{N}} (Z_n, z_n)$ and there exists a homomorphism $(\tilde{X}, \tilde{x}_0) \xrightarrow{\pi} (X, x_0)$ with $\pi^{-1}(x_0) = \{\tilde{x}_0\}$.*

PROOF. Let $\{U_n\}_{n \in \mathbf{N}}$ be a basis for the topology of X such that $\partial U_n \cap \mathcal{O}(x_0) = \emptyset$ $\forall n \in \mathbf{N}$. Let $f_n(k) = F_n(T^k x_0)$ where F_n is the characteristic function of U_n. Consider f_n as an element of Ω_2 and put $(\bar{\mathcal{O}}(f_n), f_n) = (Z_n, z_n)$, $(\tilde{X}, \tilde{x}_0) = \vee_{n \in \mathbf{N}} (Z_n, z_n)$.

Define $\pi : \tilde{X} \to X$ as follows: given $\tilde{x} \in \tilde{X}$ there exists a sequence $\{n_i\}$ with $\lim T^{n_i} \tilde{x}_0 = \tilde{x}$, let $\pi(\tilde{x}) = \lim T^{n_i} x_0$. We have to show that (i) the limit exists and (ii) the definition is independent of the particular sequence $\{n_i\}$. We shall prove (i) and (ii) by showing that whenever $\{n_i'\}$ and $\{n_i''\}$ are sequences such that

$$\lim T^{n_i'} \tilde{x}_0 = \tilde{x}, \qquad \lim T^{n_i'} x_0 = x'$$

and

$$\lim T^{n_i''} \tilde{x}_0 = \tilde{x}, \qquad \lim T^{n_i''} x_0 = x''$$

then $x' = x''$. Suppose $x' \neq x''$, then there exists U_n with $F_n(x') = 1$, $F_n(x'') = 0$ and such that x' and x'' are continuity points of F_n. Then

$$1 = F_n(x') = \lim F_n(T^{n_i'} x_0) = \lim f_n(n_i') = \tilde{x}_{(n)}(0)$$

and also

$$0 = F_n(x'') = \lim F_n(T^{n_i''} x_i) = \lim f_n(n_i'') = \tilde{x}_{(n)}(0)$$

(where $\tilde{x}_{(n)}$ is the projection of $\tilde{x}$ on Z_n); this is a contradiction. Thus $x' = x''$ and π is well defined. Clearly, $\pi(\tilde{x}_0) = x_0$ and $\pi(T\tilde{x}) = T\pi(\tilde{x})$ for every $\tilde{x} \in \tilde{X}$.

Next we check the continuity of π. Let $\tilde{x} = \lim \tilde{x}_n$ in $\tilde{X}$, and write $x_n = \pi(\tilde{x}_n)$, $x = \pi(\tilde{x})$. Suppose for some subsequence $\lim x_{n_i} = x'$. By definition of π we can choose m_i such that $T^{m_i}\tilde{x}_0$ is close enough to $\tilde{x}_{n_i}$ and $T^{m_i}x_0$ is close enough to $x_{n_i} = \pi(\tilde{x}_{n_i})$ so that $\lim T^{m_i}\tilde{x}_0 = \tilde{x}$ and $\lim T^{m_i}x_0 = x'$. But the definition of π also implies that $\lim T^{m_i}x_0 = x = \pi(\tilde{x})$ so that $x = x'$ and π is continuous.

Let $\tilde{x} \in X$ with $\pi(\tilde{x}) = x_0$. Choose n_i such that $\lim T^{n_i}\tilde{x}_o = \tilde{x}$, then for every j and k, by continuity of F_j at $T^k x_0$, we have

$$\tilde{x}_{(j)}(k) = \lim F_j(T^{k+n_i}x_0) = F_j(T^k x_0) = \tilde{x}_{0(j)}(k),$$

i.e. $\tilde{x} = \tilde{x}_0$. This shows that $\pi^{-1}(x_0) = \{\tilde{x}_0\}$ and an easy argument now implies the minimality of $\tilde{X} = \bar{\mathcal{O}}(\tilde{x}_0)$. The proof is complete. □

LEMMA 2.3. *There exist pointed minimal subflows* (X_α, x_α) *of* Ω_2 *such that* $|\mathfrak{A}| = \bigvee (X_\alpha, x_\alpha)$, *and hence* $|\mathfrak{A}| = \bigvee (X_\beta, x_\beta)$ *where* (X_β, x_β) *ranges over all minimal subflows of* Ω_2.

PROOF. For every $f \in \mathcal{M}$ let (X_f, f) be the pointed metric minimal flow $(\bar{\mathcal{O}}(f), f)$ where $\bar{\mathcal{O}}(f) \subset [-\|f\|, \|f\|]^{\mathbf{Z}}$. Clearly $|\mathfrak{A}| = \bigvee \{(X_f, f) : f \in \mathcal{M}\}$. By Lemma 2.2, there exist minimal almost one-to-one extensions $(\tilde{X}_f, \tilde{f}) \xrightarrow{\pi} (X_f, f)$ where for each $f \in \mathcal{M}$, $\tilde{X}_f = \bigvee_{i \in \mathbf{N}} Z_{f,i}$ and $Z_{f,i} \subset \Omega_2$. Let $\mathfrak{A}'$ be the algebra which corresponds to $\bigvee_{f \in \mathcal{M}} \bigvee_{i \in \mathbf{N}} Z_{f,i} = \bigvee_{f \in \mathcal{M}} \tilde{X}_f$; then $\mathfrak{A}' = \mathfrak{A}$ and we conclude that $|\mathfrak{A}| = \bigvee_{f \in \mathcal{M}} \bigvee_{i \in \mathbf{N}} Z_{f,i}$. □

LEMMA 2.4. *Every function in* $\mathfrak{A}$ *whose range is in* $\{0, 1\}$ *is an element of* R_0.

PROOF. Let $f : \mathbf{Z} \to \{0, 1\}$ be in $\mathfrak{A}$, let $F \in C(|\mathfrak{A}|)$ with $f(n) = F(T^n x_0)$. Put

$$U_\varepsilon = \{x \in |\mathfrak{A}| : F(x) = \varepsilon\} \quad (\varepsilon = 0, 1).$$

By Lemma 2.3, $|\mathfrak{A}| = \bigvee (X_\alpha, x_\alpha)$ for some family of minimal subflows of Ω_2. Since U_0 and U_1 are clopen sets F depends only on finitely many coordinates, say $\alpha_1, \cdots, \alpha_N$. Thus we can consider F as a continuous function on $\bigvee_{j=1}^N (X_{\alpha_j}, x_{\alpha_j}) = (X, x_0)$ with $f(n) = F(T^n x_0)\ \forall n \in \mathbf{Z}$ and thus $f \in R_0$. □

PROOF OF THEOREM 1(1). Suppose $A \in \mathcal{I}_{\mathfrak{A}}$ but not small. Define $\gamma \in \Omega_* = \{0, *\}^{\mathbf{Z}}$ by

$$\gamma(n) = \begin{cases} * & n \in A, \\ 0 & n \notin A. \end{cases}$$

Since A is not small we can find $\xi \in \bar{\mathcal{O}}(\gamma)$ such that ξ is minimal and $\not\equiv 0$. Let n_i be a sequence with $\lim \sigma^{n_i}\gamma = \xi$. Let $B_n = [\xi(-n), \cdots, \xi(n)]$ be the sequence of central blocks of ξ. By the minimality of ξ, passing to some subsequence of n_i, we can find a sequence $m_i \nearrow \infty$ such that:

(1) In B_{m_i} there are $2^{2m_{i-1}+1}$ disjoint appearances of $B_{m_{i-1}}$.

(2) The intervals $[n_i - m_i, n_i + m_i]$ are disjoint.

(3) $\sigma^{n_i}\gamma \mid [-m_i, m_i] = B_{m_i} = \xi \mid [-m_i, m_i]$.

By induction we define an element $\eta \in \Omega_2$ as follows. Let $\eta \mid [n_1 - m_1, n_1 + m_1]$ be identically zero. Suppose η has already been defined on $I_{i-1} = [n_{i-1} - m_{i-1}, n_{i-1} + m_{i-1}]$; we next define η on $I_i = [n_i - m_i, n_i + m_i]$. Consider B_{m_i}; we are going to change all $*$'s in B_{m_i} into zeros and ones (however, we never change zeros). The central $(2m_{i-1}+1)$-sub-block of B_{m_i} we change into $\eta \mid [n_{i-1} - m_{i-1}, n_{i-1} + m_{i-1}]$. (By induction hypothesis this does not change zeros into ones.)

There are now at least $2^{2m_{i-1}+1} - 1$ disjoint appearances of $B_{m_{i-1}}$ left in B_{m_i}. If there are r_i $*$'s in $B_{m_{i-1}}$ $(r_i \leqq 2m_{i-1}+1)$, then there are 2^{r_i} possible replacements of stars into zeros and ones; and we put all these replacements in the $2^{2m_{i-1}}+1$ disjoint appearances of $B_{m_{i-1}}$ in B_{m_i}. On the rest of B_{m_i} we now replace all $*$'s by zeros. Let $\tilde{B}_{m_i}$ be the new block of 0 and 1 thus obtained, then define $\eta \mid I_i = \tilde{B}_{m_i}$. This defines η on $I = \bigcup_{i=1}^{\infty} I_i$. Define η on $\mathbf{Z} \setminus I$ to be identically zero. We clearly have $\lim \sigma^{n_i}\eta = \theta$ for some $\theta \in \Omega_2$.

Next we show that $\theta \in \mathfrak{A}$. Since A is an $\mathfrak{A}$-interpolation set, there exists $f \in \mathfrak{A}$ with $f \mid A = \eta \mid A$. By passing to a subsequence and then relabelling, we can assume that $\lim T^{n_i} f = g$ exists and by Lemma 2.1 $g \in \mathfrak{A}$. Put $D = \{n \in \mathbf{Z} : \xi(n) = *\}$; then 1_D is a minimal function (hence in $\mathfrak{A}$) and recalling that the support of η, i.e. the set $\{n : \eta(n) = 1\}$, is contained in A, we have

$$\theta = \lim \sigma^{n_i}\eta = \lim \sigma^{n_i}(1_A \cdot \eta) = \lim T^{n_i}(1_A \cdot f) = 1_D \cdot g.$$

Hence $\theta \in \mathfrak{A}$. By Lemma 2.4 $\theta \in R_0$ and we conclude that $X = \bar{\mathcal{O}}(\theta)$ is a restricted subset of Ω_2. Define

$$Y_0 = \{y \in \Omega_2 : y \mid D \equiv 0\}$$

and let Y be the smallest closed invariant subset of Ω_2 containing Y_0. Since 1_D is minimal and $D \neq \emptyset$ it is clear that $Y \neq \Omega_2$. On the other hand, the construction of θ ensures that for every finite subset $\{k_1, \cdots, k_n\}$ of D and a sequence $\varepsilon_1, \cdots, \varepsilon_n$ where $\varepsilon_i = 0$ or 1, there exists an m with $T^m\theta(k_i) = \varepsilon_i$, $i = 1, \cdots, n$. It follows that $X + Y = \Omega_2$. This contradiction shows that $A \in \mathcal{I}_{\mathfrak{A}}$ can not be small and the proof is complete. □

REMARK. Let $\mathcal{A}$ be an algebra of minimal functions. We can see directly that $\mathcal{I}_{\mathcal{A}}$ consists of small sets. In fact it is an easy exercise to see that if A is not small and (X, x_0) is a pointed minimal flow, then

$$V = \operatorname{int}\overline{\{T^n_{x_0} : n \in A\}} \neq \emptyset.$$

Therefore, there exists $k \in A$ with $T^k x_0 \in V$ and if X is infinite, we can conclude that $\{T^n x_0\}_{n \in A}$ accumulates at $T^k x_0$. Applying this result to the minimal flow $(|\mathscr{A}|, x_0)$, we immediately see that A can not be an $\mathscr{A}$-interpolation set.

PROOF OF THEOREM 1(2). Let $A \subset \mathbf{Z}$ be a small set. Again we use $\Omega_* = \{0,*\}^{\mathbf{Z}}$ and consider $\xi \in \Omega_*$ defined by

$$\xi(n) = \begin{cases} * & n \in \mathbf{Z} \setminus A, \\ 0 & n \in A. \end{cases}$$

The fact that A is small means that for every k there exists an N_k, such that in any block of length N_k in ξ, there is a sub-block of k consecutive $*$'s.

For a block B we let $|B|$ be its length. We are going to distinguish certain sub-blocks of ξ consisting of only stars, which will be called niches of order 1, 2, 3, etc. A pairwise disjoint subfamily of niches, called the skeleton $\mathscr{S}$, will be further distinguished with the following properties:

(i) All n-niches have a common length t_n.

(ii) For every n the gap between any niche in $\mathscr{S}$ of order $\geqq n$ and the next niche in $\mathscr{S}$ with order $\geqq n$ is $\leqq 2N_{t_n}$.

We shall distinguish as well a sequence of blocks $B_2, B_3, \cdots$ of ξ such that:

(a) The domain of B_n contains the interval $[-N_{t_{n-1}}, N_{t_{n-1}}]$.

(b) $|B_n| = t_n$.

(c) B_n begins and ends with $(n-1)$-niches belonging to the skeleton.

Note that these conditions imply that the n-niches of the skeleton are disjoint from B_n.

Once these constructions are accomplished we can conclude the proof of the theorem as follows.

Let there be given an arbitrary function $\varphi : A \to \{0,1\}$. We define an element $\xi_\varphi \in \{0,1\}^{\mathbf{Z}}$ with the property that $\xi_\varphi \mid A = \varphi$ and such that ξ_φ is a minimal function.

Our first step is to define ξ_φ on the domain of B_2. We set $\xi_\varphi(k) = \varphi(k)$ if $\xi(k) = 0$ (i.e. if $k \in A$) and otherwise our choice is arbitrary, say $\xi_\varphi(k) = 1$ for all other k in the domain of B_2.

In any niche of order two in $\mathscr{S}$ we define ξ_φ to coincide with its definition on B_2.

Next we define ξ_φ on what is left of the domain of B_3. Whenever $\xi(k) = 0$ we let $\xi_\varphi(k) = \varphi(k)$. On the domain of B_2 and on those niches of order 2 in $\mathscr{S}$ which are contained in B_2, ξ_φ is already defined; and we define it arbitrarily on the rest of the domain of B_3, for example, by putting 1 everywhere. On every niche of order 3 in $\mathcal{S}$ we let ξ_φ coincide with its values on B_3, etc. Since the union of the domains of the B_n's is all of $\mathbf{Z}$, this inductive definition yields an element $\xi_\varphi \in \Omega_2$.

By induction from B_i to B_{i+1}, starting with B_{n+1} and using properties (ii) and (c), we see that the block — $\xi_\varphi \mid$ domain of B_n — appears in ξ_φ with gaps $\leqq 2N_{t_n}$. This proves the minimality of ξ_φ.

Moreover, since the skeleton structure is independent of φ, it is clear that for every $\varepsilon > 0$ the set

$$\{n \in \mathbf{Z} : d(\sigma^n \xi_\varphi, \xi_\varphi) < \varepsilon,\ \forall \varphi \in \{0,1\}^A\}$$

is syndetic; i.e. the point $x_0 = (\xi_\varphi)_{\varphi \in \{0,1\}^A}$ is an almost periodic point of the pointed flow $(X, x_0) = \bigvee \{(\bar{\mathcal{O}}(\xi_\varphi), \xi_\varphi) : \varphi \in \{0,1\}^A\}$. This means that (X, x_0) is a minimal flow and the corresponding algebra $\mathcal{A}$ clearly contains the family $\{\xi_\varphi\}_{\varphi \in \{0,1\}^A}$. It follows that $A \in \mathcal{I}_{\mathcal{A}}$, and our proof will be complete when the construction of the skeleton of niches and the sequence $B_2, B_3, \cdots$ is accomplished.

We now turn to these constructions.

Niches of order 1. These are all blocks of stars of length one in ξ. We denote this family of niches by $\mathcal{N}_1$.

Niches of order 2 *and the block* B_2. Let B_2 be a sub-block of ξ whose domain contains the interval $[-N_1, N_1]$ which begins and ends with a 1-niche (i.e. with stars). We let our second order niches (or 2-niches) be disjoint blocks in ξ of length $|B_2| = t_2$ of consecutive stars, disjoint from B_2, and such that every block of length N_{t_2} in ξ contains at least one such block. We denote the family of 2-niches by $\mathcal{N}_2$.

Suppose that B_k and the family $\mathcal{N}_k$ of niches of order k have already been constructed for $k \leqq n$.

Niches of order n and the block B_{n+1}. Let B_{n+1} be a sub-block of ξ whose domain contains the interval $[-N_{t_n}, N_{t_n}]$ (where $t_n = |B_n|$), which begins and ends with n-niches. We let our $(n+1)$-niches be disjoint blocks of $t_{n+1} = |B_{n+1}|$ consecutive stars disjoint from B_{n+1}, such that every block of length $N_{t_{n+1}}$ in ξ contains at least one such block. We let $\mathcal{N}_{n+1}$ be the set of $(n+1)$-niches.

This completes the inductive construction of niches of all orders.

Next we define a triangular array $\mathcal{N}_n^{(k)}$ $(k \geqq n)$ of families of niches. For every n we put $\mathcal{N}_n^{(n)} = \mathcal{N}_n$.

Suppose

$$
\begin{array}{ccccc}
\mathcal{N}_1^{(1)} & \mathcal{N}_1^{(2)} & \cdots & \mathcal{N}_1^{(n-1)} & \mathcal{N}_1^{(n)} \\
 & \mathcal{N}_2^{(2)} & \cdots & \mathcal{N}_2^{(n-1)} & \mathcal{N}_2^{(n)} \\
 & & & \vdots & \vdots \\
 & & & \mathcal{N}_{n-1}^{(n-1)} & \mathcal{N}_{n-1}^{(n)} \\
 & & & & \mathcal{N}_n^{(n)}
\end{array}
$$

have been defined; we now describe $\mathcal{N}_n^{(n+1)}, \mathcal{N}_{n-1}^{(n+1)}, \cdots, \mathcal{N}_2^{(n+1)}, \mathcal{N}_1^{(n+1)}$. $\mathcal{N}_n^{(n+1)}$ is obtained from $\mathcal{N}_n^{(n)}$ by the omission from $\mathcal{N}_n^{(n)}$ of every niche which is either contained in or intersects an element of $\mathcal{N}_{n+1}^{(n+1)}$. $\mathcal{N}_{n-1}^{(n+1)}$ is obtained from $\mathcal{N}_{n-1}^{(n-1)}$ by the omission from $\mathcal{N}_{n-1}^{(n-1)}$ of every niche which is either contained in or intersects a niche in $\mathcal{N}_{n+1}^{(n+1)} \cup \mathcal{N}_n^{(n+1)}$. Suppose $\mathcal{N}_k^{(n+1)}$ has been defined for $l < k \leqq n+1$; we let $\mathcal{N}_l^{(n+1)}$ consist of all niches in $\mathcal{N}_l^{(l)}$ which are neither contained in nor intersect niches in $\bigcup_{n+1 \geqq k > l} \mathcal{N}_k^{(n+1)}$.

This completes our inductive construction of the triangular array.

To define the skeleton $\mathcal{S}$, we want to check that on the domain of B_n, $\mathcal{N}_j^{(m)} = \mathcal{N}_j^{(n)}$ for all $m \geqq n$, and all $j \leqq n-1$. This is easily seen by a descending induction on j from $n-1$ down to 1 using the fact that no niche of order $\geqq n$ can intersect the domain of B_n.

For each n let $\mathcal{S}_n$ be the set of niches in $\bigcup_{j=1}^{n-1} \mathcal{N}_j^{(n)}$ which lie in the domain of B_n and let $\mathcal{S} = \bigcup_{n=2}^{\infty} \mathcal{S}_n$.

Clearly our skeleton consists of pairwise disjoint niches. Property (ii) of $\mathcal{S}$ follows from the fact that it holds in each $\mathcal{S}_k$. This completes the proof of Theorem 1(2).

Part (3) of Theorem 1 now follows from parts (1) and (2). In fact, the proof of part (2) shows that the small sets are interpolation sets for the collection $\mathcal{M}$ of minimal functions in l^∞, which generates $\mathfrak{A}$. Part (1) shows that only small sets can be in $\mathcal{I}_{\mathfrak{A}}$.

§3. Ideals and interpolation sets

We recall some elementary facts about $\beta\mathbf{Z}$. One can view the elements of $\beta\mathbf{Z}$ as ultrafilters on $\mathbf{Z}$ where $n \in \mathbf{Z}$ is identified with the ultrafilter $\{A \subset \mathbf{Z} : n \in A\}$. Given $A \subset \mathbf{Z}$ its closure in $\beta\mathbf{Z}$ is given by $\bar{A} = \{p \in \beta\mathbf{Z} : A \in p\}$. The sets $\{\bar{A} : A \subset \mathbf{Z}\}$ form a basis for open sets in $\beta\mathbf{Z}$ and $\bar{A} \cap \mathbf{Z} = A$.

Every map $\varphi : \mathbf{Z} \to X$ where X is a compact Hunsdorff space can be continuously extended to $\beta\mathbf{Z}$. In particular an $f \in l^\infty(\mathbf{Z})$ can be uniquely extended to $\beta\mathbf{Z}$. We shall identify f with its extension. In terms of this identification it is easy to see that $A \subset \mathbf{Z}$ is an $\mathcal{A}$-interpolation set iff the functions of $\mathcal{A}$ separate points of $\bar{A} \subset \beta\mathbf{Z}$.

The map $n \to n+1$ of $\mathbf{Z}$ into $\beta\mathbf{Z}$ can likewise be extended to a map, which we denote by T, of $\beta\mathbf{Z}$ into $\beta\mathbf{Z}$. Clearly $(\beta\mathbf{Z}, T, 0)$ is a pointed flow and, in fact, it is a universal object for pointed flows. In a similar way, addition in $\mathbf{Z}$ can be extended to "addition" in $\beta\mathbf{Z}$ which makes $\beta\mathbf{Z}$ a semigroup. For details we refer to [1], [5] and [6].

PROOF OF THEOREM 2 PARTS (1) AND (2). It directly follows from the definition of K and $\mathfrak{B}(K)$ that $\mathcal{I} \subset \mathcal{I}_{\mathfrak{B}(K)}$.

If A is a $\mathfrak{B}(K)$-interpolation set but $A \notin \mathcal{I}$, then $\bar{A} \cap K \neq \emptyset$. We now show that $\bar{A} \cap K$ is a singleton. Suppose $p, q \in \bar{A} \cap K$ and $p \neq q$. Choose $B_0, B_1 \subset \mathbf{Z}$ such that $\bar{B}_0$ and $\bar{B}_1$ are disjoint neighborhoods of p and q, respectively. We define $\varphi : A \to \{0,1\}$ by $\varphi \mid B_0 \cap A \equiv 0$ and $\varphi \mid B_1 \cap A \equiv 1$ and on the rest of A, φ is arbitrary, say 1. We can find $f \in \mathfrak{B}(K)$ with $f \mid A = \varphi$. But then $f(p) = 0$ and $f(q) = 1$ contradicts the fact that functions in $\mathfrak{B}(K)$ do not distinguish between points of K.

Thus $\mathcal{I} \subset \mathcal{I}_{\mathfrak{B}(K)}$ and if $\mathcal{I} \neq \mathcal{I}_{\mathfrak{B}(K)}$, then K has an isolated point, since $\bar{A}$ is clopen.

We now assume that K has an isolated point, say $\{p\} = \bar{A} \cap K$ for some $A \subset \mathbf{Z}$. We shall show that $\mathcal{I}$ cannot be the ideal of $\mathcal{A}$-interpolation sets for any algebra $\mathcal{A}$. Assume $\mathcal{I} = \mathcal{I}_{\mathcal{A}}$, then since $A \notin \mathcal{I}$ we also have $A \notin \mathcal{I}_{\mathcal{A}}$. This means that there are points $q_1, q_2 \in \bar{A}$ such that $q_1 \neq q_2$ and $f(q_1) = f(q_2)$ for every $f \in \mathcal{A}$. If both $q_1 \neq p$ and $q_2 \neq p$, then we can find basic disjoint neighborhoods $\bar{B}_1$ and $\bar{B}_2$ of q_1 and q_2 which are disjoint from K. Then $B = B_1 \cup B_2 \in \mathcal{I}$ and hence $B \in \mathcal{I}_{\mathcal{A}}$. Define $\varphi : B \to \{0,1\}$ by $\varphi \mid B_1 \equiv 1$, $\varphi \mid B_2 \equiv 0$ and choose $f \in \mathcal{A}$ with $f \mid B = \varphi$, then $f(q_1) = 1$ and $f(q_2) = 0$, a contradiction.

Thus we can assume that there is at most one point $q \in \bar{A}$ such that $p \neq q$, $\mathcal{A}$ does not distinguish between p and q and such that $\mathcal{A}$ separates all other points in $\bar{A}$. Let $\bar{B}$ be a basic neighborhood of q disjoint from K, and let $A' = A \setminus B$. Then A' is an $\mathcal{A}$-interpolation set ($\mathcal{A}$ separates points on $\bar{A}'$) and $A' \notin \mathcal{I}$ because $p \in \bar{A}' \cap K$. This contradiction completes the proof.

We complete the proof of Theorem 2 with an example of an invariant ideal for which K contains an isolated point.

Let $A = \{n_i\}_{i=1}^{\infty}$ be a sequence such that $n_i \to \infty$ and $n_{i+1} - n_i \to \infty$. Let p be an ultrafilter on $\mathbf{Z}$ containing A ($p \in \bar{A}$). Put

$$\mathcal{F} = \{B \subset \mathbf{Z} \colon \exists F \in p \text{ and } \exists \text{ a function } k : F \to \mathbf{N}; \; f \to k_f, \text{ such that } k_f \to \infty \text{ when } |f| \to \infty \text{ and } \forall f \in F \; [f - k_f, f + k_f] \subset B\}.$$

If B_1 and B_2 are in $\mathcal{F}$, F_1, F_2, $k^{(1)}$ and $k^{(2)}$, the corresponding sets and functions, we put $F = F_1 \cap F_2$, $k_f = \min(k_f^{(1)}, k_f^{(2)})$ and then

$$B_1 \cap B_2 \supset B = \bigcup_{f \in F} [f - k_f, f + k_f] \quad \text{and} \quad B_1 \cap B_2 \in \mathcal{F}.$$

Thus $\mathcal{F}$ is a filter. Given $B \in \mathcal{F}$ and an integer l, we find a set $F \in p$ and a function k_f on F corresponding to B and then for large $|f|$, we have

$$B + l \supset [f - k_f + |l|, f + k_f - |l|].$$

Thus $\mathcal{F}$ is invariant and clearly $\mathcal{F} \subset p$ (i.e. $p \in K = \bigcap \{\bar{F} : F \in \mathcal{F}\}$).

Suppose now that $q \in K \cap \bar{A}$ and let $F \in p$. If $F \notin q$, then $A \setminus (F \cap A) \in q$. We can find a function $k : F \cap A \to \mathbf{N}$, $f \to k_f$ and $k_f \nearrow \infty$ such that $F \cap A = B \cap A$ where $B = \bigcup_{f \in F \cap A} [f - k_f, f + k_f]$.

Now by definition $B \in \mathcal{F}$, hence

$$\emptyset = B \cap [A \setminus (F \cap A)] = B \cap [A \setminus (B \cap A)] \in q;$$

a contradiction. Thus $p \subset q$ and hence $p = q$, i.e., we have shown that $K \cap \bar{A} = \{p\}$ and p is an isolated point of K. □

We now turn to the proof of the corollary to Theorem 2. It is shown in [6] that for the ideals described in (1), (2) and (3) of the corollary, the corresponding kernels K are the sets:

$K_1 = \text{closure } (\bigcup \{M : M \text{ is a minimal ideal in } \beta\mathbf{Z}\})$,

$K_2 = M$, a minimal ideal in $\beta\mathbf{Z}$,

$K_3 = \text{closure } (\bigcup \{\text{Supp}\mu : \mu \text{ an invariant probability measure on } \beta\mathbf{Z}\})$,

respectively.

All we have to show is that none of these sets can have an isolated point. For K_2 this is clear because M is a minimal set of the flow $(\beta\mathbf{Z}, T)$. Similarly, if $A \subset \mathbf{Z}$ and $\bar{A} \cap K_1 \neq \emptyset$, then since $\bar{A}$ is open, we have for some M, minimal ideal, $\emptyset \neq M \cap \bar{A} \supset \bar{A} \cap K_1$ and again the same observation applies. Finally if $\bar{A} \cap K_3 \neq \emptyset$ then $\emptyset \neq \bar{A} \cap L \subset \bar{A} \cap K_3$ for some closed invariant subset L which is the support of some invariant probability measure μ on $\beta\mathbf{Z}$. But then $\mu(\bar{A} \cap L) > 0$, and in particular $\bar{A} \cap L$ cannot be a singleton. □

Proof of Theorem 3. (1) Suppose first that $A \in \mathcal{S}(\mathcal{A})$. Let $n \in \text{cls}_{J_{\mathcal{A}}} A$ (the $J_{\mathcal{A}}$-closure of A); i.e. in the pointed flow $(|\mathcal{A}|, T, x_0)$ we have $T^n x_0 \in \overline{\{T^k x_0 : k \in A\}}$. Then $x_0 \in \overline{\{T^{k-n} x_0 : k \in A\}}$ and since $(\overline{A - n}) \cap K = \emptyset$, this implies that $0 \in \overline{A - n}$, i.e. $n \in A$, and A is $J_{\mathcal{A}}$ closed.

If $n \in A$ let $B = A \setminus \{n\}$, then $B \in \mathcal{S}(\mathcal{A})$, hence B is $J_{\mathcal{A}}$ closed, in particular $n \notin \mathrm{cls}_{J_{\mathcal{A}}} B$ and n is an isolated point of A in the relative $J_{\mathcal{A}}$ topology. Thus A is $J_{\mathcal{A}}$ discrete.

Conversely, if $A \subset \mathbf{Z}$ is $J_{\mathcal{A}}$ closed and discrete and for some n, $\overline{A+n} \cap K_0 \neq \emptyset$, then $T^{-n} x_0 \in \overline{\{T^k x_0 : k \in A\}}$ and $-n \in \mathrm{cls}_{J_{\mathcal{A}}} A$. Since A is $J_{\mathcal{A}}$ closed we have $-n \in A$ and this contradicts the $J_{\mathcal{A}}$ discreteness of A. Thus $A \in \mathcal{S}(\mathcal{A})$.

(2) This follows from (1) and [6, th. 5.1(1)].

(3) It was shown in [6, th. 5.1(2) and (3)] that (i) implies (ii) and that (ii) implies that every $\mathcal{A}$ interpolation set is $J_{\mathcal{A}}$ closed and discrete. Thus by (1) either (i) or (ii) implies $\mathcal{I}_{\mathcal{A}} \subset \mathcal{S}(\mathcal{A})$.

Assume (iii) and let $A \in \mathcal{I}_{\mathcal{A}}$. Then $\bar{A} \cap K_0$ is not a singleton. If $\bar{A} \cap K_0$ is not empty we can find disjoint sets $A_0, A_1 \subset \mathbf{Z}$ with $A = A_0 \cup A_1$ and such that $\bar{A}_0 \cap K \neq \emptyset$ and $\bar{A}_1 \cap K \neq \emptyset$. Define $\varphi : A \to \{0,1\}$ by $\varphi \mid A_0 \equiv 0$, $\varphi \mid A_1 \equiv 1$. Let $f \in \mathcal{A}$ with $f \mid A = \varphi$, then f assumes two different values on K_0 in contradiction to the fact that $f \in \mathcal{A}$.

(4) We merely have to notice that the filter $\mathcal{F}$ and, therefore, the set K_0 are the same whether we take our algebra to be $\mathcal{A}$ or $\tilde{\mathcal{A}}$ because $J_{\mathcal{A}} = J_{\tilde{\mathcal{A}}}$. Thus $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\tilde{\mathcal{A}})$. Also by [8] if $\mathcal{A}$ is Souslin so is $\tilde{\mathcal{A}}$.

This completes the proof of Theorem 3. □

Before proving Theorem 4 we recall the following definitions and results from [4] and [6] (see also [3]). As was noted before, $(\beta\mathbf{Z}, T, 0)$ is the universal pointed point transitive flow; i.e. for every pointed flow (X, x_0) there exists a unique homomorphism $\varphi : \beta\mathbf{Z} \to \overline{\mathcal{O}(x_0)}$ with $\varphi(0) = x_0$. We write $\varphi(p) = px_0$, and one can think of px_0 as the limit of the ultrafilter $\{\{T^n x_0\}_{n \in A}\}_{A \in p}$. This defines an "action" of the semigroup $\beta\mathbf{Z}$ on any flow X; for $p, q \in \beta\mathbf{Z}$ and $x \in X$ we have $p(qx) = (pq)x$. If u is an idempotent of $\beta\mathbf{Z}$ then ux is always proximal to x, since $u(ux) = ux$.

For $A \subset \mathbf{Z}$ and $p \in \beta\mathbf{Z}$ we write $p * A = \{n \in \mathbf{Z} : T^n p \in \bar{A}\} = \{n \in \mathbf{Z} : p \in \overline{A - n}\}$. It is easy to check that when 1_A is considered as a point of the flow (Ω_2, σ) then $1_{p*A} = p1_A$.

Let $\{i_n\}_{n=1}^{\infty}$ be a sequence in $\mathbf{Z}$, for any finite subset α of $\mathbf{N}$ we let $i_\alpha = \sum_{n \in \alpha} i_n$ and we write $\mathrm{IP}(\{i_n\}_{n=1}^{\infty}) = \{i_\alpha : \alpha$ a finite subset of $\mathbf{N}\}$. $\mathrm{IP}(\{i_n\}_{n=1}^{\infty})$ is called an IP-system, and a subset $A \subset \mathbf{Z}$ is called an IP-set if it contains an infinite IP-system. A set $A \subset \mathbf{Z}$ is an IP-set iff $\bar{A}$ in $\beta\mathbf{Z}$ contains an idempotent $\neq 0$ ([6]); $A \subset \mathbf{Z}$ is called an MIP set if $\bar{A}$ contains a minimal idempotent, i.e. an idempotent which belongs to some minimal ideal. A is an MIP set iff there exists a subset $B \subset \mathbf{Z}$ such that, $0 \in B$, 1_B is a minimal function and 1_A is proximal to 1_B in (Ω_2, σ) (iff A is a central set in the sense of [3]).

Finally if (X, T) is a flow and $\mathrm{IP}(\{i_n\}_{n=1}^{\infty}) = \{i_\alpha\}$ is an IP-system, then we write IP-lim $T^{i_\alpha}x = y$ for $x, y \in X$, if for every neighborhood V of y there exists an N such that for every α, a finite subset of $\{N+1, N+2, \cdots\}$, $T^{i_\alpha}x \in V$.

Proof of Theorem 4. (1) We first notice that if x is a recurrent point of a flow (X, T), i.e. if there exists a sequence $\{n_i\}$ with $|n_i| \nearrow \infty$ such that $\lim T^{n_i}x = x$, then for some subsequence $\{n_{i_j}\}$ the corresponding IP-system $\mathrm{IP}\{n_{i_j}\} = \{n_\alpha\}$ satisfies IP-lim $T^{n_\alpha}x = x$.

Now a subset $A \subset \mathbf{Z}$ is an IP-set iff there exists an idempotent $u \neq 0$ in $\bar{A}$, iff $0 \in u * A$. Let $u * A = B$, then in (Ω_2, α), $u 1_A = u 1_B = 1_B$. Thus, 1_A is proximal to 1_B which is fixed under u and satisfies $1_B(0) = 1$.

Conversely if $A \subset \mathbf{Z}$ is such that for some sequence n_i ($|n_i| \nearrow \infty$), $\lim \sigma^{n_i} 1_A = 1_B = \lim \sigma^{n_i} 1_B$ and $1_B(0) = 1$, then we also have IP-lim $\sigma^{n_\alpha} 1_A = 1_B =$ IP-lim $\sigma^{n_\alpha} 1_B$ for some IP-system $\{n_\alpha\} = \mathrm{IP}(\{n_{i_j}\})$, generated by a subsequence $\{n_{i_j}\}$ of $\{n_i\}$. But then for some N and every finite set $\alpha \subset \{N+1, N+2, \cdots\}$

$$\sigma^{n_\alpha} 1_A(0) = 1_A(n_\alpha) = 1_B(0) = 1 \quad \text{and} \quad n_\alpha \in A.$$

So that A is an IP-set.

Write

$$\begin{aligned} W &= \{(\xi, \eta) \in \Omega_2 \times \Omega_2 : \exists n_i \text{ with } |n_i| \nearrow \infty \text{ and} \\ &\qquad \lim \sigma^{n_i}\xi = \xi = \lim \sigma^{n_i}\eta\,;\ \xi(0) = 1\} \\ &= \bigcap_{n \in \mathbf{N}} \bigcup_{|k| > n} \bigcap_{|j| \leq n} \{(\xi, \eta) : \xi(k+j) = \xi(j) = \eta(k+j);\ \xi(0) = 1\}. \end{aligned}$$

Then W is a Borel subset of $\Omega_2 \times \Omega_2$ and $\pi_2 W = \{1_A : A \text{ is an IP-set}\}$ is Souslin.

(2) The same proof will work for

$$\begin{aligned} W &= \{(\xi, \eta) : \xi \text{ and } \eta \text{ are proximal, } \xi \text{ is minimal and } \xi(0) = 1\} \\ &= \bigcap_{n \in \mathbf{N}} \bigcup_{d \in \mathbf{N}} \bigcap_{l \in \mathbf{Z}} \bigcup_{0 \leq m \leq d} \bigcup_{k \in \mathbf{Z}} \bigcap_{|j| \leq n} \{(\xi, \eta) : \xi(l - m + j) = \\ &\qquad \xi(j) = \xi(j + k) = \eta(j + k);\ \xi(0) = 1\}. \end{aligned}$$

□

We conclude with an example of an algebra $\mathcal{A} \subset l^\infty$ such that $\mathcal{A} \neq l^\infty$ and such that there are $A, B \in \mathcal{I}_{\mathcal{A}}$ with $\mathbf{Z} = A \cup B$. (In the terminology of [6] the collection $\mathcal{P} = \{A \subset \mathbf{Z} : A \notin \mathcal{I}_{\mathcal{A}}\}$ is not quasidivisible.)

Put

$$A = \bigcup_{n=1}^{\infty} [2^n, 2^n + 2^{n-1} - 1], \qquad B = \bigcup_{n=1}^{\infty} [2^n + 2^{n-1}, 2^{n+1} - 1],$$

and let

$$\mathscr{A}_0 = \{f \in l^\infty : \exists N \text{ s.t. for } n > N \text{ and } N < j < 2^{n-1} - N,$$
$$f(2^n + j) = f(2^n + 2^{n-1} + j)\}.$$

Clearly $\mathscr{A}_0$ is a translation invariant subalgebra of l^∞ containing the constant functions. Let $\mathscr{A}$ be the uniform closure of $\mathscr{A}_0$ in l^∞. Then $\mathscr{A}$ is a proper subalgebra of l^∞ and $A, B \in \mathscr{I}_{\mathscr{A}}$.

PROBLEMS. (A) For a minimal idempotent v let $\mathfrak{A}(v)$ be the corresponding maximal algebra of minimal functions. Is $\mathscr{I}_{\mathfrak{A}(v)}$ an ideal? Is this the ideal of small sets?

(B) Let $\mathscr{L} = \bigcap \{\mathfrak{A}(v) : v \in J\}$ (where J is the set of idempotents in a fixed minimal left ideal in $\beta \mathbf{Z}$); this is the algebra of point distal functions. Is $\mathscr{I}_{\mathscr{L}}$ an ideal?

As was shown in [6] $A \in \mathscr{I}_{\mathscr{L}}$ cannot contain an infinite IP-set. Does $\mathscr{I}_{\mathscr{L}}$ coincide with the ideal of sets A that do not contain a translate of an infinite IP-set (or an MIP-set)? Can the latter ideal be represented as $\mathscr{I}_{\mathscr{A}}$ for some algebra $\mathscr{A}$?

(C) In the content of Theorem 3, is the condition "K_0 contains no isolated points" necessary for the conclusion that $\mathscr{I}_{\tilde{\mathscr{A}}} = \mathscr{S}(\tilde{\mathscr{A}})$?

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